Series: 7, 8, 9, 10, 11.

7.

1). $\sum_{n=0}^{+\infty} \frac{n+1}{3^n}$

$$\int_0^x \sum_{n=0}^{+\infty} (n+1) t^n \, dt = \frac{x}{1-x} \Rightarrow \sum_{n=0}^{+\infty} (n+1)x^n = \left(\frac{x}{1-x}\right)' = \frac{1-x-x\cdot(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$$

$$\Rightarrow \sum_{n=0}^{+\infty} \frac{n+1}{3^n} = \frac{1}{(1-\frac{1}{3})^2} = \frac{9}{4}$$

2). $\sum_{n=3}^{+\infty} \frac{2n-1}{n^3-4n}$

$$\frac{2n-1}{n^3-4n} = \frac{2n-1}{(n-2)n(n+2)} = \frac{2}{(n-2)(n+2)} - \frac{1}{(n-2)n(n+2)}$$

$$= \frac{1}{2}\left(\frac{1}{n-2} - \frac{1}{n+2}\right) - \frac{1}{2}\left(\frac{1}{n-2} - \frac{1}{n}\right)\frac{1}{n+2}$$

$$= \frac{1}{2}\left(\frac{1}{n-2} - \frac{1}{n+2}\right) - \frac{1}{8}\cdot\frac{1}{n-2} + \frac{1}{8}\cdot\frac{1}{n+2} + \frac{1}{4}\cdot\frac{1}{n} - \frac{1}{4}\cdot\frac{1}{n+2}$$

$$= \frac{3}{8}\cdot\frac{1}{n-2} + \frac{1}{4}\cdot\frac{1}{n} - \frac{5}{8}\cdot\frac{1}{n+2}$$

$$= \frac{3}{8}\cdot\left(\frac{1}{n-2} - \frac{1}{n}\right) + \frac{5}{8}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\sum_{n=3}^{+\infty} \frac{2n-1}{n^3-4n} = \frac{3}{8}\sum_{n=3}^{+\infty} \frac{1}{n-2} - \frac{1}{n} + \frac{5}{8}\sum_{n=3}^{+\infty} \frac{1}{n} - \frac{1}{n+2}$$

$$= \frac{3}{8}\left(1+\frac{1}{2}\right) + \frac{5}{8}\left(\frac{1}{3}+\frac{1}{4}\right)$$

$$= \frac{9}{16} + \frac{35}{96} = \frac{54+35}{96} = \frac{89}{96}$$

3). $$\sum_{n=1}^{+\infty} \frac{n^2 x^{n-2}}{(n-1)!} = \sum_{n=1}^{+\infty} \frac{n\cdot(n-1)x^{n-2}}{(n-1)!} + \sum_{n=1}^{+\infty} \frac{n x^{n-2}}{(n-1)!}$$

$$= \left(\sum_{n=1}^{+\infty} \frac{x^n}{(n-1)!}\right)' + \frac{1}{x}\left(\sum_{n=1}^{+\infty} \frac{x^n}{(n-1)!}\right)'$$

$$= (x\cdot e^x)' + \frac{1}{x}(x\cdot e^x)'$$

$$= (x+1)e^x\cdot\left(1+\frac{1}{x}\right) = \frac{(x+1)^2 e^x}{x}$$

$$\Rightarrow \sum_{n=1}^{+\infty} \frac{n^2}{(n-1)!} = 2^2\cdot e = 4e$$

4). $$\sum_{n=2}^{+\infty}\left(\frac{1}{\sqrt{n-1}} + \frac{1}{\sqrt{n+1}} - \frac{2}{\sqrt{n}}\right) = \sum_{n=2}^{+\infty}\left(\frac{1}{\sqrt{n-1}} - \frac{1}{\sqrt{n}}\right) + \sum_{n=2}^{+\infty}\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n}}\right) = 1 - \frac{1}{\sqrt{2}}$$

5). $\ln\left(1+\frac{(-1)^{2k}}{2k}\right)+\ln\left(1+\frac{(-1)^{2k+1}}{2k+1}\right)$

$= \ln\frac{2k+1}{2k} + \ln\frac{2k}{2k+1} = 0$

$\Rightarrow \sum_{n=2}^{2k+1} \ln\left(1+\frac{(-1)^n}{n}\right) = 0$

$\sum_{n=2}^{2k} \ln\left(1+\frac{(-1)^n}{n}\right) = \ln\frac{2k+1}{2k}$

$\Rightarrow \sum_{n=2}^{+\infty} \ln\left(1+\frac{(-1)^n}{n}\right) = 0$

6). $\sum_{n=0}^{+\infty} \ln\left(\cos\frac{\alpha}{2^n}\right) \quad \alpha \in]0, \frac{\pi}{2}]$

$\sum_{n=0}^{k} \ln\left(\cos\frac{\alpha}{2^n}\right) + \ln\sin\frac{\alpha}{2^k}$

$= \sum_{n=0}^{k-1} \ln\cos\frac{\alpha}{2^n} + \ln\sin\frac{\alpha}{2^{k-1}} - \ln 2$

$= \ln\sin(2\alpha) - \ln 2^{k+1}$

$\Rightarrow \sum_{n=0}^{k} \ln\left(\cos\frac{\alpha}{2^n}\right) = \ln\sin 2\alpha - \ln\left(2^{k+1}\sin\frac{\alpha}{2^k}\right)$

$\Rightarrow \sum_{n=0}^{+\infty} \ln\left(\cos\frac{\alpha}{2^n}\right) = \ln\sin 2\alpha - \ln(2\alpha) = \ln\frac{\sin 2\alpha}{2\alpha}$

7). $\sum_{n=0}^{+\infty} \frac{\operatorname{th}\frac{a}{2^n}}{2^n}, \quad a \in \mathbb{R}^*$

$\int_0^x \frac{\operatorname{th}\frac{t}{2^n}}{2^n}\,dt = \ln\left(\operatorname{ch}\frac{x}{2^n}\right) \Rightarrow \int_0^x \sum_{n=0}^{+\infty} \frac{\operatorname{th}\frac{t}{2^n}}{2^n}\,dt = \sum_{n=0}^{+\infty} \ln\left(\operatorname{ch}\frac{x}{2^n}\right)$

$\sum_{n=0}^{k} \ln\left(\operatorname{ch}\frac{x}{2^n}\right) + \ln\left(\operatorname{sh}\frac{x}{2^k}\right)$

$= \ln(\operatorname{sh}(2x)) - \ln 2^{k+1}$

$\Rightarrow \sum_{n=0}^{+\infty} \ln\left(\operatorname{ch}\frac{x}{2^n}\right) = \ln(\operatorname{sh}(2x)) - \lim_{k\to+\infty} \ln\left(2^{k+1}\cdot \operatorname{sh}\frac{x}{2^k}\right) = \ln\frac{\operatorname{sh}(2x)}{2x}$

$\Rightarrow \sum_{n=0}^{+\infty} \frac{\operatorname{th}\frac{a}{2^n}}{2^n} = \left(\ln\frac{\operatorname{sh}(2x)}{2x}\right)'\Big|_{x=a}$

$= \left.\frac{\frac{2\operatorname{ch}(2x)\cdot 2x - 2\cdot\operatorname{sh}(2x)}{4x^2}}{\frac{\operatorname{sh}(2x)}{2x}}\right|_{x=a} = \frac{2x\cdot\operatorname{ch}(2a) - \operatorname{sh}(2a)}{a\cdot\operatorname{sh}(2a)}$

8. $\sum_{k=1}^{n} u_n > n\cdot u_n$

$\Rightarrow \lim_{n\to+\infty} n\cdot u_n = 0 \Rightarrow u_n = o(\frac{1}{n})$

Example: $u_n = \begin{cases} 0 & n \neq k^2 \\ \frac{1}{k^2} & n = k^2 \end{cases}$

9. $a_n = \left(e - \sum_{k=0}^{n} \frac{1}{k!}\right)\cdot(n+1)!$

$= \sum_{k=n+1}^{+\infty} \frac{1}{k!}\cdot(n+1)!$

$= 1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)} + \sum_{k=5}^{+\infty} \frac{1}{(n+2)(n+3)\cdots(n+1+k)}$

where $\lim_{n\to+\infty} \sum_{k=5}^{+\infty} \frac{n^4}{(n+2)(n+3)\cdots(n+1+k)}$

$= \lim_{n\to+\infty} \frac{n^4}{(n+2)(n+3)(n+4)(n+5)} \sum_{k=1}^{+\infty} \frac{1}{(n+5+1)\cdots(n+5+k)}$

$= \lim_{n\to+\infty} \sum_{k=1}^{+\infty} \frac{1}{(n+5+1)\cdots(n+5+k)} < \lim_{n\to+\infty} \sum_{k=1}^{+\infty} \frac{1}{(n+6)^k} = \lim_{n\to\infty} \frac{\frac{1}{n+6}}{1-\frac{1}{n+6}} = 0$

$\Rightarrow \quad a_n$

$= 1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)} + o(\frac{1}{n^4})$

$= 1 + \frac{1}{n} + (\frac{1}{n+2} - \frac{1}{n}) + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)} + o(\frac{1}{n^4})$

$= 1 + \frac{1}{n} - \frac{1}{n^2} + \left(\frac{1}{(n+2)(n+3)(n+4)} - \frac{1}{n(n+2)(n+3)}\right) + \left(\frac{6}{n^2(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)}\right) + o(\frac{1}{n^4})$

$= 1 + \frac{1}{n} - \frac{1}{n^2} + \frac{-4}{n(n+2)(n+3)(n+4)} + \frac{6}{n^2(n+2)(n+3)} + \frac{1}{(n+2)(n+3)(n+4)(n+5)} + o(\frac{1}{n^4})$

$= 1 + \frac{1}{n} - \frac{1}{n^2} + \frac{3}{n^4} + o(\frac{1}{n^4})$

10. $u_n = \sin(\pi(2+\sqrt{3})^n)$ (黄星皓)

Since $(2+\sqrt{3})^n + (2-\sqrt{3})^n$

$= \sum_{k=0}^{n} C_n^k \left(2^{n-k}\cdot\sqrt{3}^k + 2^{n-k}\cdot(-\sqrt{3})^k\right)$

$= 2\sum_{k \text{ is even}}^{n} C_n^k \cdot 2^{n-k}\cdot(\sqrt{3})^k$

$= 2\sum_{l=0}^{[\frac{n}{2}]} C_n^{2l}\cdot 2^{n-2l}\cdot 3^l$ is even.

then

$$\sin(\pi(2+\sqrt{3})^n)$$

$$= -\sin(\pi(2-\sqrt{3})^n).$$

$$(2-\sqrt{3})<1 \Rightarrow \left|\sum_{n=0}^{+\infty}\sin(\pi(2+\sqrt{3})^n)\right|$$

$$< \sum_{n=0}^{+\infty}\pi\cdot(2-\sqrt{3})^n < +\infty.$$

11. $$\left(\sum_{k=1}^{n}\frac{\sqrt{u_n}}{n}\right)^2$$

$$\leq \left(\sum_{k=1}^{n}u_k\right)\cdot\left(\sum_{k=1}^{n}\frac{1}{k^2}\right) \to \frac{\pi^2}{6}\cdot\sum_{k=1}^{+\infty}u_k < +\infty$$